MULTIDISCIPLINARY RESEARCH

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A Critical Study of Zeta Function and Riemann Hypothesis from Various Fields of Mathematics

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Abstract:

The aim of this paper to study the relation between the zeta function and the primes extends beyond equation. A hypothesis due to Riemann regarding the zeros of zeta, and hence it is known as the Riemann hypothesis and has many implications about the distribution of the primes, as well as better approximations to different arithmetic functions such as the difference between the logarithmic integral Li(x) and the prime counting function $\pi(x)$. We study the zeta function and the Riemann Hypothesis from various perspectives and fields of mathematics such as number theory, complex analysis, and random matrices. Here, we would like to introduce a new approach 1 which simplifies the study of RH from the complex plane to a real line.

KeyWords : Zeta function, Prime counting function, Riemann Hypothesis, Dirichlet L-series,

Introduction:

We can find various results from definition of Zeta function given by Euler. We have investigated a lot of fruitful results from these workouts. It is found very interesting, convenient and worthy to use in designing the data mining models for very complex data and spatial data like DNA bank, finger prints, foot prints and other geographical data.

Euler's zeta function is defined for any real number, say 's' greater than 1 by the infinite sum as following.

$$\varsigma(s) = 1 + \frac{1}{n^3} \quad \text{......................} \quad (1)$$

Where n is natural number and s is any bigger number than 1 [8].

We are first going to look the derived results from Euler's function. Then we will constitute the model to use this result in meta-querying to access the discovered information from giant database.

Let us start the thing by deriving the harmonic series simply given as following.

1 + 1/2 + 1/3 + ¼ + 1/5 + 1/5 + 1/7 + ... (2)

Now, Can we guess the result of sum of the above harmonic series? As the numbers of terms in the series is although infinite but the range of sum for the solution is rather finite. The result can be probably estimated with some limit value.

This feature we will use as indexes described in the previous chapter of this thesis. The indexes further helps in the algorithm to find the data in complex database.

We can apply the Euler's theorem further by dividing the above harmonic series in 6.2 as sum of prime harmonic terms and some other terms shown as following.

S1 = 1+ ½ + 1/3 + 1/5 +1/7 + 1/11 + 1/13 + 1/17 + 1/19 + ... (3)

Other remaining terms are below.

S2 = ¼ + 1/6 + 1/8 + 1/9 + 1/10 .... (4)

The value of prime harmonic series of equation 6.3 can be directly evaluated from Euler's function and the same can be done in multiple fashions for equation 6.4.

For an example a lot of terms of equation 6.4 can be re-written as following.

¼ + 1/6 + 1/8 + 1/10 ..... = (1/2)*(½ + 1/3 + ¼ + 1/5 .....) ..... (5)

The idea is divide the huge infinite series into few series of "Prime Harmonics", which can be very easily evaluated by Euler's formulae [3-6].

The similar efforts we do in computing the parameters like indexes, supports, weights, range etc in making discovery in spatial databases. The found pattern can be used by neural networks in order to do the search efficient and faster with a lot of ease.

Now let us define the complete Euler's function in order to enhance and elaborate the above concept in wider applications.

Basics of Zeta Function :

After Euler defined this function, he showed that it has a deep and profound connection with the pattern of the primes. We are providing here the definitions of Zeta functions given by Euler.

$$\zeta(s) = \pi\left(\frac{1}{1+n^3}\right) \qquad \text{.........} \qquad (6)$$

Where n are the prime numbers and s is any real number greater than 1.

The above definition indicates that the Euler's function can be observed as multiple of prime terms.

Now we are focusing the result derived from the above study like Euler's discovery to write the following formulae.

$$(1-x)^{-1} = 1/(1-x)$$
$$= 1+x+x^2+x^3+x^4+ \qquad \text{...........} \qquad (7)$$

So, we can write,

$$\frac{1}{1+n^3} = 1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}} \qquad \text{............} \qquad (8)$$

This expression on the left is a typical term in Euler's infinite product, of course, so the above equation provides an infinite sum expression for each term in the infinite product. What Euler did next was multiply together all of these infinite sums to give an alternative expression for his infinite product.

Using the ordinary algebraic rules for multiplying (a finite number of finite) sums, but applying them this time to an infinite number of infinite sums, you see that when you write out the right-hand side as a single infinite sum, its terms are all the expressions of the product form.

Now, from the point of view of the subsequent development of mathematics it was not so much the fact that the prime harmonic series has an infinite sum that is important, even though it did provide a completely new proof of Euclid's result that there are infinitely many primes[11-14]. Rather, Euler's infinite product formula for *æ*(*s*) marked the beginning of analytic number theory.

In 1837, the French mathematician Lejeune Dirichlet generalized Euler's method to prove that in any arithmetic progression *a, a+k, a+2k, a+3k, . . .,* where *a* and *k* have no common factor, there are infinitely many primes. Euclid's theorem can be regarded as the special case of this for the arithmetic progression 1, 3, 5, 7, . of all odd numbers.

The principal modification to Euler's method that Dirichlet made was to modify the zeta function so that the primes were separated into separate categories, depending on the remainder they left when divided by *k*. His modified zeta function had the form.

$$L(s, \chi) = \chi(1)/1^s + \chi(2)/2^s + \chi(3)/3^s + \quad \ldots\ldots\ldots\ldots\ldots \qquad (9)$$

where $\chi(n)$ is a special kind of function - which Dirichlet called a "character" that splits the primes up in the required way. In particular, it must be the case that $\chi(mn) = \chi(m)\chi(n)$ for any *m, n*. The other conditions are that $\chi(n)$ depends only on the remainder you get when you divide *n* by *k*, and that $\chi(n) = 0$ if *n* and *k* have a common factor[1-2].

Any function of the form $L(s,\chi)$ where *s* is a real number greater than 1 and χ is a character is known as a Dirichlet L-series. The Riemann zeta function is the special case that arises when you take $\chi(n) = 1$ for all *n*.

Findings of Zeta Functions:

Mathematicians subsequent to Dirichlet generalized the notion by allowing the variable *s* and the numbers $\chi(n)$ to be complex numbers, and used the generalized versions to prove a great many results about prime numbers, thereby demonstrating that the L-series provide an extremely powerful tool for the study of the primes.

A key result about L-functions is that, as with the zeta function, they can be expressed as an infinite product over the prime numbers (sometimes known as an Euler product), namely.

$$L(s, \chi) = \pi\left(\frac{1}{1-\frac{\chi(p)}{p^s}}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Where p is the prime terms and s is any real number > 1.

There are a lot of proofs to use the above Euler's results but we are highlighting some of them following.

(i) Cauchy's derivation of integration

(ii) Hadamard's Integration

(iii) First evaluation integration

(iv) Second evaluation integration

(v) Edward's Expansion

(vi) Riemann's Hypothesis

Mining Data by using Zeta Function:

We use the zeta function as evaluator of the weights like mapper, reducer and support parameters to allocate the database resource, search from local and global catalog, perform shuffling, transfer and provide the solution of the meta-query asked by the clients.

We are listing the flow chart for data mining process for the same.

First we initialize the system and then perform the following steps consecutively.

Step 1: Allocation of the data or information in the storage block.

Step 2: Mapper produces the join key and record pairs.

Step 3: Each mapper split and processes one block.

Step 4: The processed blocks are shuffled and sorted in the global network.

Step 5: Reducer performs the actual join of the databases

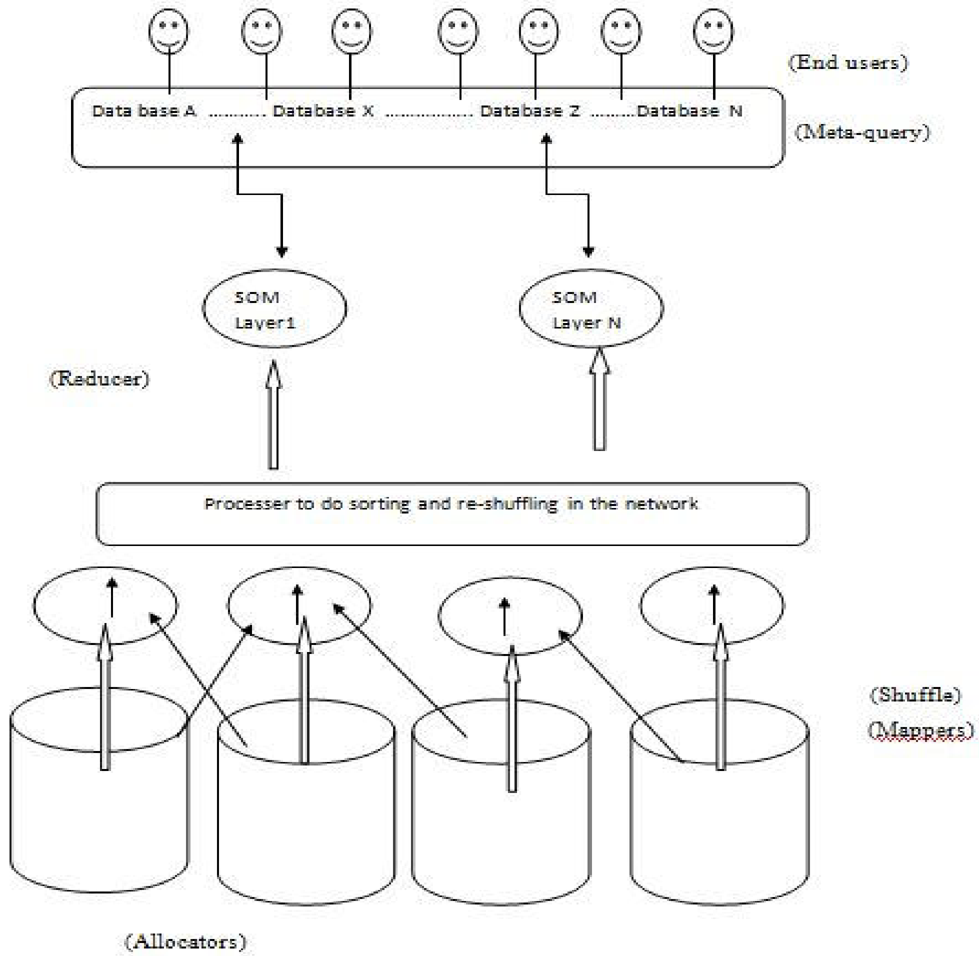
Step 6: Indexes and weighted self-organized map mine the required information.

Step 7: Meta-querying layer represents the discovered information.

Step 8: Un-link the resources.

Step 9: Stop the data mining process.

The layout of the schematic flow chart of the above data mining process by joining the databases is depicted in the following sketch diagram. The model has the architecture of several processes done in the proposed system and how it is working.



(Figure 1: Data mining model architecture)

Theory and Properties of Zeta:

Riemann pointed out that the zeta function has two types of zeros. First trivial zeros that consist of all negative even integers. Second, an infinite number of non-trivial zeros which are all complex, and are known to lie in the strip 0 < ® (s) < 1. The trivial zeros are well understood, but the study of the non-trivial zeros of zeta is still ongoing.

The Riemann zeta function can be given in various equivalent forms. In the right half-plane σ > 0, the zeta function can be defined as follows.

$$2\ \mathrm{SIN}(\pi s)\ \pi(s-1)\ \pi(s) = \int \frac{(-x)^{s-1}}{e^x - 1}\,dx$$

This formula gives a good approximation for the zeta function in the critical strip, and can thus be used in the study of the Riemann Hypothesis. The uniqueness obtained from the principle of analytic continuation guarantees that this definition is consistent with equation.

The Zeta function satisfies an important property of reflexivity. This property we will use in mining data and information and to control duplicity in the databases.

We see that mathematicians First, by fixing the imaginary part of arbitrary zeros of the zeta function, and thus studying the real part of those zeros along a horizontal line. Second, we can study the imaginary part of the zeros of zeta on the critical line, and search for embedded patterns and relations amongst them.

This provides valuable information on the distribution of the zeros of the zeta function on the critical line. We have listed the following R-H rules generated for data mining tools.

- Correlation Coefficient (CC) = $\dfrac{(n-1)\sum_{i-1}^{n}(p_i-\bar{p})(a_i-\bar{a})}{\left[\sum_{i-1}^{n}(p_i-\bar{p})^2\right]\left[\sum_{i-1}^{n}(a_i-\bar{a})^2\right]}$ (Rule 1)

- Mean Absolute Error (MAE) $= \dfrac{\sum_{i=1}^{n}|p_i-a_i|}{n}$ (Rule 2)

- Root Mean –Squared Error (RMSE) $= \sqrt{\dfrac{\sum_{i=1}^{n}(p_i-a_i)^2}{n}}$ (Rule 3)

- Relative Absolute Error (RAE) $= \sqrt{\dfrac{\sum_{i=1}^{n}|p_i-a_i|}{\sum_{i=1}^{n}\left|a_i-\bar{a}\right|}}$ (Rule 4)

- Root Relative Squared Error (RRSE) $= \sqrt{\dfrac{\sum_{i=1}^{n}(p_i-a_i)^2}{\sum_{i=1}^{n}\left(a_i-\bar{a}\right)^2}}$ (Rule 5)

The non-trivial zeros of the Riemann zeta function lie on the critical line <(s) or = 1. Riemann first stated this hypothesis in his paper "On the Number of Prime Numbers less than a Given Quantity" that was published in 1859. Mathematicians have since been struggling to prove the hypothesis.

This includes Riemann himself, who admits to having given up, at least temporarily, on a proof of RH after several unsuccessful attempts [13-15].

Riemann conjectured that all the non-trivial zeros of zeta lie on this critical line. This is known as the Riemann Hypothesis (RH), considered to be one of the most important open problems in mathematics. Due to its importance, the Clay Institute of Mathematics is offering a $1 million prize for the first person to prove the hypothesis [14].

Since all the non-trivial zeros of the zeta function calculated thus far have real part equal to ½, it suffices to only study their imaginary parts.

We study the zeros of zeta on the critical line by studying the sequence given by definition 8. It is important to note that by n, we mean a positive integer, and that we use the notation $\{\alpha_n\}_n$ for countable sequences of real numbers α_n.

Conclusion:

Euler thought of splitting this sum into two parts, a sum of all the terms involving the primes, and a sum involving the terms with composite numbers. He wanted to show that the latter sum is convergent, and thus conclude that the sum of the reciprocals of all primes diverges. Yet, since it is infinite, and its two parts do not both converge, Euler was not able to split the harmonic series the way he wanted. That is, an infinite series cannot be split into various parts unless all the parts converge.

An alternate was to study the Dirichlet L_{+1} series $= \sum \frac{1}{n^s}$ starting with the divergent harmonic series as s approaches one from the right. This series converges as long as the single complex variable s is strictly larger than one. Hence, it can be split into the two sums the way Euler desired. The zeta function also appears in physics; especially in areas relevant to chaos in classical and quantum mechanics. Here in our proposed research we are using to retrieve data and information from complex pattern like special databases and Data ware House. The discovery process for spatial data is more complex than for relational data. This applies to both the efficiency of algorithms as well to the complexity of possible patterns that can be found in a spatial database.

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